B.Sc. 3rd Semester (Honours) Examination, 2019-20

MATHEMATICS

Course ID : 32111 **Course Code : SHMTH-301-C-5**

Course Title: Theory of Real Functions and Introduction to Metric Spaces

Time: 2 Hours **Full Marks: 40**

The figures in the right hand side margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Unless otherwise mentioned, symbols have their usual meaning.

1. Answer *any five* questions: 2×5=10

(a) Show that $\underset{x\to 0}{\text{Lt}}\frac{1}{x}\sin\frac{1}{x}$ does not exist.

(b) Prove or disprove: The intermediate value theorem is applicable to the function $f(x) = \begin{cases} 2x+1, & x \in (0,1] \\ 0, & x = 0 \end{cases}$.

(c) Show that on the real numbers with the usual metric, the set of natural numbers is closed.

(d) Test if Lagrange's mean value theorem holds for the function $f(x) = |x|$ in the interval $[-1, 1]$.

(e) Prove or disprove:

If $f(x)$ and $g(x)$ be two functions such that $\lim_{x\to a}[f(x)+g(x)]$ exists, then $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

(f) Show that the function $f(x) = \cos\frac{1}{x}, 0 < x < 1$, is not uniformly continuous on $(0, 1)$.

(g) Find the diameter of the set

$\left\{(x, y): 0 < x < \frac{\pi}{2},\ y = \cos x\right\}$ with respect to usual metric on R^2.

(h) Show that in a discrete metric space (X, d), every subset of X is open set.

2. Answer *any four* questions: 5×4=20

(a) (i) Prove that $\underset{x\to a}{\text{Lt}} f(x)$ exists and is equal to l if and only if $\underset{x\to a^+}{\text{Lt}} f(x)$ and $\underset{x\to a^-}{\text{Lt}} f(x)$ both exist and are equal to l.

(ii) If a function f is derivable in a closed interval $[a, b]$ and $f'(a) \neq f'(b)$, and k is a real number lying between $f'(a)$ and $f'(b)$, then show that there exists at least one point $c \in (a, b)$ such that $f'(c) = k$. 3+2=5

(b) If f' exists and is bounded on some interval I, then prove that f is uniformly continuous on I.

(c) Let (X, d) be a metric space and ρ be a function on X x X defined by $\rho(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in x$. Show that ρ is a metric on x.

(d) (i) Let $f: [a, b] \to R$ be a function and $c \in [a, b]$ and for every sequence $\{x_n\}$ in $[a, b]$ which converges to 'c', we have $\lim_{n \to \alpha} f(x_n) = f(c)$, then show that $f(x)$ is continuous at $x = c$.

(ii) Write $\in -\delta$ definition of a function not to be uniformly continuous. 3+2=5

(e) (i) A function $f: R \to R$ is continuous on R. Prove that the set $S = \{x \in R: f(x) > 0\}$ is an open set in R, where R is the set of Reals.

(ii) Prove or disprove:

Every continuous function is always monotonic. 3+2=5

(f) (i) Show that a subset A of a metric space (X, d) is closed if and only if $X - A$ is open set.

(ii) In mean value theorem,

$f(h) = f(0) + h\, f'(\theta h), 0 < \theta < 1, \lim_{h \to 0} \theta = \frac{1}{2}$, when $f(x) = \cos x$.

3. Answer *any one* question: 10×1=10

(a) (i) Let $f(x) = \begin{cases} x^p \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ Obtain p such that $f(x)$ is continuous at $x = 0$ and $f(x)$ is differentiable at $x = 0$.

(ii) Let f be continuous on $[a - h, a + h]$ and derivable on $(a - h,\ a + h)$. Prove that there exists a real number θ $(0 < \theta < 1)$ for which

$f(a + h) - 2\, f(a) + f(a - h) = h\, [f'(a + \theta h) - f'(a - \theta h)]$.

(iii) Give an example with justification to show that an open set may not be an open sphere.

5+3+2=10

(b) (i) When is a function $f(x)$ said to have local maxima at $x = a$? Does $f'(a) = 0$ always imply existence of an extremum of f at $x = c$? Justify.

(ii) Expand $\sin\theta$ as a finite series of expansion in 'θ'.

(iii) When a function f is said to be convex function on an interval $[a, b]$? If f is convex in $[a, b]$, then show that f'' is non-negative in $[a, b]$.

(iv) Let (X, d) be a metric space and let $x, y (\neq x) \in X$. Prove that there is a nbd. M of x and a nbd N of y such that $M \cap N = \varphi$. 2+2+4+2=10

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